**About infinity: excursus to the past.**

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**Annotation.**

Two concepts of infinity are known in the history of mathematics, these are Aristotelian and Cantorian. The last one was formulated by the author of the set theory G. Cantor about one and half century ago, and at the present time this concept is dominating. To elaborate his concept, Cantor used the so-called **diagonal method** for comparison of the cardinality of sets of infinite series with that of the natural series as much as the **Cantor's theorem** about prevalence of the cardinality of the set of all subsets of any set A over the cardinality of A itself. In the present work it is shown by use of specific examples (i.e., 'constructively') that arguments used by Cantor are not quite rigorous and consequently the concept of unique **potential infinity** seems to be more acceptable.

*Millions of years will pass before we'll be able to understand why we tend to cognize infinity.*
P. Erdös.

## Introduction.

The mathematics is rested on three pillars: zero, unit and infinity symbolized respectively as 0, 1 and ∞. Their existence is usually postulated in the systems of axioms of logics and arithmetic's. These notions are very capacious, intimately interrelated and go out far beyond the mathematics. The unit may be enlarged or be subdivided as much as desired. It is a seed of the natural number sequence emerging as a result of successive summation of unit with itself. The notion of ***mathematical infinity*** is used to symbolize the ***potential*** (***virtual***) ***result*** of this unlimited process. Ciphers 0 and 1 are enough to write down any natural number in binary number system:

$$a = a_0 2^0 + a_1 2^1 + ... + a_i 2^i + ... \equiv a_0 a_1 ... a_i ... \qquad (1)$$

and any proper fraction as dyadic expansion:

$$0.a \equiv 0.a_0 a_1 a_2 ... a_n ... = a_0 \frac{1}{2^1} + a_1 \frac{1}{2^2} + a_2 \frac{1}{2^3} + ... + a_{n-1} \frac{1}{2^n} + ..., \qquad (2)$$

where each $a_n$ is equal to 0 or 1. For finite natural numbers the series (1) terminates at some finite index $n$, that is $a_n = 1$, $a_{n+1} = a_{n+2} = … = 0$. The number $a$ is usually 'read' as follows: $a = 1a_{n-1}a_{n-2}...a_0$. The corresponding 'finite' fraction $0.a_0a_1…a_{n-1}1$ may be represented as 'true infinite' fraction according to the rule:

$$0.a_0a_1 \ ... \ a_{n-1}1000… = 0.a_0a_1 \ ... \ a_{n-1}\,0111… \qquad (3)$$

Each fraction may be considered as a point of the segment [0, 1], and the set of all fractions has a cardinality of the continuum $c$ in the present-day treatment. Note also, that the set of infinite sequences $a \equiv a_0a_1…a_i...$ may be symbolized as $V^\infty$, where V is the two-element set {0, 1}.

The notion *infinity*, widely used also outside the mathematics, is an item of perpetual discussions in the scientific community and within the general public. In the science it was thought since Aristotelian times that "infinity is always in the possibility and not in the actuality". K. Gauss wrote in 1831: "I protest against the use of an infinity quantity as something completed; which is never permissible in mathematics". This tradition was broken with by the founder of the set theory G. Cantor, who introduced the notion of the completed, 'actual' infinity as opposed to the Aristotelian 'potential infinity'. Cantor generalized the notion of the ***number of elements*** to the infinite sets by using the term ***cardinality*** (or ***cardinal number***) of the set and the symbol ℵ (*aleph*) to represent cardinalities of the infinite sets.

According to Cantor ***there exist*** (in the same sense as usual numbers) infinite sets with different cardinal numbers. The sequence of the cardinal numbers of the infinite sets in the order of their growth looks like

$$\aleph_0<\aleph_1<\aleph_2<\ldots, \tag{4}$$

where the minimal cardinal number $\aleph_0$ is a cardinality of the natural number sequence **N**. The series (4) must contain the cardinality of continuum *c*, which, according to Cantor, exceeds $\aleph_0$: $c > \aleph_0$. The assumption $c = \aleph_1$ is called the ***continuum hypothesis***.

Such radical intrusion to the foundations of mathematics met rather ambiguous responses from the prominent scientists of the end of 19-th and the beginning of 20-th centuries. Some contradictions (paradoxes) of the ***Cantor's set theory*** were pointed out, and they were thought to be resolvable by the improvement of the logic of the mathematical reasoning. The efforts in this direction resulted in the foundation and development of ***mathematical logic***. The proposed aim of this mathematical branch was the complete formalization of the processes of inference and proof (***Hilbert's program***), which would allow to avoid errors and contradictions. The essential restrictions on the feasibility of this program were imposed by famous ***Gödel's theorems*** about consistency and incompleteness of different systems of axioms. These theorems encourage the reconsideration of some nontransparent aspects of the set theory, such as, for instance, the system of alephs (4), which was characterized by H. Weyl as "the mist on the mist".

The above-mentioned relation $c > \aleph_0$ takes an important position in the theory of the cardinal numbers. In this work we present some reasons in favor of alternative relation, $c = \aleph_0$, that is, the power of continuum coincides with the cardinality of the natural number sequence. As a matter of fact, this is the return from the ***Cantor's concept*** of infinity to ***Aristotelian***, or rather the erasing the boundary between them.

Some support in favor of the unique infinity is provided by ***Cantor's very definition*** of the set as "a collection of ***definite, distinguishable objects*** of our perception or our thought conceived as a whole". The collection of 'distinguishable' objects may in principle be dissociated into elements and renumbered, that is, this set is ***denumerable*** (finite or countable). Or, in other words, for any two infinite Cantor's sets one may initiate a ***potentially infinite*** process of composing of pairs of elements *resulting* in the ***one-to-one correspondence*** of these two sets.

Not every collection of objects satisfies Cantor's definition of sets. For instance, since the creation of quantum mechanics the sets of identical particles which obey to the quantum-mechanical ***concept of particle indistinguishability*** have come into the scientific practice. Some restrictions on the quantitative characteristics of the quantum-mechanical sets were also imposed by the Heisenberg's principle of uncertainty. In quantum theory the notion of ***Observer*** is explicitly introduced, who regulates the measurement processes, and this is in harmony with inclusion of human ***perception and thought*** into the definition of sets.

The seeming paradoxicalness of the equality of continuum and natural number series cardinalities is evidently connected with the representation of these sets as a segment of the number axis and isolated points on this axis correspondingly. However, if one performs the following ***potentially infinite*** procedure on the segment: to throw out the middle one third of the initial segment, then the middle thirds of two remained parts of the initial segment and so on, then as a 'result' the so-called triadic set of Cantor (*Cantor's dust*) emerges. An ***appearance*** of this set is like an infinite collection of points, but it has a cardinality of continuum.

In the first part of this essay, we consider formal aspects of some simple infinite 'universal' table (UT), the separate fragments of which may serve as tables of truth functions of arbitrary number of arguments in the statement calculus – the introductory chapter of mathematical logic. This UT allows us to establish in a natural way the correspondence between the natural number sequence and the set of dyadic expansions (2), which ***potentially*** seems to be one-to-one. In the second part we discuss some consequences of the return to

Aristotelian standpoint, one of which is a certain symmetry between properties of naught (zero) and infinity.

## 1. Truth functions in the statement calculus.

The aforementioned infinite table UT may be constructed formally without using logic terminology (table 1). Let us write down natural numbers in order as headings of the table columns. In column *a* we write digits $a_i$ used in the dyadic representation (1) of the number *a*.

Table 1. Natural numbers in dyadic calculus.

| | **0** | **1** | **2** | **3** | **…** | ***a*** | **…** |
|---|---|---|---|---|---|---|---|
| **0** | 0 | 1 | 0 | 1 | … | $a_0$ | … |
| **1** | 0 | 0 | 1 | 1 | … | $a_1$ | … |
| **2** | 0 | 0 | 0 | 0 | … | $a_2$ | … |
| **…** | … | … | … | … | … | … | … |
| ***i*** | 0 | 0 | 0 | 0 | … | $a_i$ | … |
| **…** | … | … | … | … | … | … | … |

The set of digits ($a_0$, $a_1$, … $a_i$, …) may be referred not only to the natural number *a* (eq. (1)), but also to the dyadic fraction 0.*a* (2), presented as an infinite series, the consequent partial sums of which form the ***Cauchy sequence***. The columns, considered as numbers, are arranged in increasing order from left to right, while the corresponding order of fractions 0.*a* seems at first to be chaotic.   The set of ***all*** fractions (2) includes (potentially) ***all*** real numbers of the segment [0, 1], so we (*preliminarily*) conclude that the sets of natural numbers and real numbers within the segment [0, 1] have equal cardinalities. This conclusion will be discussed in more details in the next section.

The interesting property of the table 1 is the periodicity of its rows. The zeroth row is periodical with the period of two: 0101… ≡ $(01)^\infty$, the first row has the period of 4: 00110011… ≡ $(0^2 1^2)^\infty$, the *i*-th row has the period of $2^{i+1}$: $\left(0^{2^i} 1^{2^i}\right)^\infty$. Therefore, if we restrict ourselves with the first *i* rows of UT, the resulting structure will be periodic exactly with the period $2^{i+1}$, so it will be fully defined by the finite table containing the first *i* + 1 rows and $2^{i+1}$ columns. This fragment of UT will be denoted as UT(*i*). The set of columns of UT(*i*) in fact coincides with the set $V^{i+1}$. This periodicity of UT allows us to conclude that it contains infinitely many Cauchy sequences with equal elements (partial sums) up to any arbitrarily high order.

Note that the tables UT(*i*) may serve as tables of the truth functions of an arbitrary number of arguments ('letters') in the statement calculus.

The '*statement*' is a declarative sentence which may be qualified as truth (T) or false (F). In other words, it is some quantity A which may take two values: 0 (T) or 1 (F). Using different sentential connectives, it is possible to produce complex statements from more primitive ones. The truth or falsity of composed statements depends on the type of the connectives and the truth or falsity of the constituents. An arbitrary statement A may be considered as a function of several ***primitive*** statements ('*letters*') $P_1$, $P_2$, …, $P_n$, which independently take values 0, 1 (T, F): A = f ($P_1$, …, $P_n$). For given values of letters $p_1$, $p_2$, …, $p_n$ ($p_i$ = 0 or 1) a function f takes the value f ($p_1$, …, $p_n$) which also equals 0 or 1. For given *n* the number of different arguments of a function f equals $2^n$ and the number of different functions f equals $2^{2^n}$. In fact, these functions map the set $V^n$ onto V, where V is the two-element set {0, 1}. The values of functions can be presented as a table with $2^n$ rows and $2^{2^n}$ columns. For *n* = 2 it is the table with 4 rows and 16 columns:

Table 2. Truth functions of two arguments.

| $P_2$ | $P_1$ | $f_0$ | $f_1$ | $f_2$ | $f_3$ | $f_4$ | $f_5$ | $f_6$ | $f_7$ | $f_8$ | $f_9$ | $f_{10}$ | $f_{11}$ | $f_{12}$ | $f_{13}$ | $f_{14}$ | $f_{15}$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 0 | 0 | 0 | 1 | 0 | 1 | 0 | 1 | 0 | 1 | 0 | 1 | 0 | 1 | 0 | 1 | 0 | 1 |
| 0 | 1 | 0 | 0 | 1 | 1 | 0 | 0 | 1 | 1 | 0 | 0 | 1 | 1 | 0 | 0 | 1 | 1 |
| 1 | 0 | 0 | 0 | 0 | 0 | 1 | 1 | 1 | 1 | 0 | 0 | 0 | 0 | 1 | 1 | 1 | 1 |
| 1 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| | | **0** | $\mid$ | $\Rightarrow$ | $\neg P_2$ | $\Rightarrow$ | $\neg P_1$ | $\Leftrightarrow$ | $\downarrow$ | $\vee$ | $\oplus$ | $P_1$ | | $P_2$ | | & | **1** |

The table is arranged as follows. The rows are numbered by pairs $i_1 i_2$, which may be considered as dyadic notation of natural numbers 0, 1, 2, 3: $i = (i_1 i_2) = i_1 \cdot 2^0 + i_2 \cdot 2^1$. Each function $f_a$ is presented as a column $a_0\, a_1\, a_2\, a_3$, corresponding to natural number $a = a_0 \cdot 2^0 + a_1 \cdot 2^1 + a_2 \cdot 2^2 + a_3 \cdot 2^3$. So, the digit $a_i$ (= 0, 1) stands in the column $a$ at $i$-th row. It is easy to see that the table 2 is a fragment UT (3) of the table 1; it consists of the elements of the first 4 rows and first 16 columns of that table.

Some logic functions have special names and symbols indicated in the lower row of the table. For instance, $f_{14}(P, Q) = P\&Q$ is a *conjunction* of statements P and Q; it is true only if both statements connected by the conjunction symbol are true. $f_8(P_1, P_2) = P_1 \vee P_2$ is *disjunction*, $f_2(P_1, P_2) = P_2 \Rightarrow P_1$, $f_4(P_1, P_2) = P_1 \Rightarrow P_2$ are *implications*, $f_3(P_1, P_2) = \neg P_2$ is *negation of* $P_2$, and so on. Note that the set of columns is closed under the operations of addition (mod 2) and multiplication, which allows us to replace logic operations by algebraic ones. We will not consider these aspects in detail since only the formal structure of tables is important for our goals.

When the number of 'letters' exceeds 2 the table of truth functions quickly becomes immense, however the principles of their formation and their structure remain the same. At arbitrary $n$ the number of different functions (table columns) is equal to $2^{2^n}$ , the height of columns (the number of rows) equals $2^n$, i.e., we deal with the fragment UT($2^n$-1) of UT. The following periodic columns serve as 'letters': $P_1 = (010101\ldots01) \equiv (01)^{2^{n-1}}$, $P_2 = (00110011\ldots0011) \equiv (0^2 1^2)^{2^{n-2}}$, $P_3 = (00001111\ldots00001111) \equiv (0^{2^2} 1^{2^2})^{2^{n-3}}$,…, $P_n = (00\ldots011\ldots1) \equiv (0^{2^{n-1}} 1^{2^{n-1}})^{2^0}$ ($2^{n-1}$ zeros and the same number of units). The function $f_a$ is a column ($a_0$, $a_1$, … $a_i$, …, $a_{2^n-1}$), where coefficients $a_i$ represent the number $a$ in dyadic calculus. Therefore, the value of the function $f_a(P_1, \ldots, P_n)$ in the row $i = (i_1 \ldots i_n) = i_1 2^0 + \ldots + i_n 2^{n-1}$ equals to $a_i$ (= 0 or 1).

The tables exhibit high symmetry. Note that the table of $n$ 'letters' many times contains tables for the numbers of letters from 1 to $n - 1$. Each row $i$ is periodic with the period $2^{i+1}$. In the first four rows the foregoing table 2 for 2 letters is periodically repeated, in the first eight rows the table 8×256 for 3 letters is repeated and so on.

As it was said earlier, the row $a = (a_0 a_1 \ldots a_{2^n - 1})$ describes not only the natural number $a$, but also the dyadic expansion

$$0.a = a_0\left(\tfrac{1}{2}\right)^1 + a_1\left(\tfrac{1}{2}\right)^2 + \ldots + a_i\left(\tfrac{1}{2}\right)^{i+1} + \ldots + a_{2^n - 1}\left(\tfrac{1}{2}\right)^{2^n}, \quad (5)$$

i.e., the set of columns is a set of fractions with denominator $2^{2^n}$ and arbitrary numerators from 0 to $2^{2^n}$ - 1. Upon unrestricted growth of $n$ the table UT($2^n$-1) tends to UT and will include any partial sum of any Cauchy sequence, that is, the whole set of the dyadic fractions, however in this process each fraction 0.*a* will have the proper ‘number’ *a*! ***Therefore, the one-to-one correspondence is ‘established’ between two sets – the set of dyadic fractions and the set of natural numbers***. According to the generally accepted agreement this means the potential equality $c = \aleph_0$ in contradiction with the Cantor’s relation $c > \aleph_0$.

## 2. Some consequences of the equality $c = \aleph_0$.

Let us discuss the possible reasons of the difference between the received above result $c = \aleph_0$ and the common Cantor’s inequality $c > \aleph_0$. Cantor proved it using so called ***diagonal method*** by direct construction of the hypothetical dyadic fraction which is not contained in any sequence of such fractions indexed by natural number system. However, this hypothetical fraction must begin as 0.1… to differ from the zero-order column, as 0.11… to differ from the first column, as 0.111… to differ from the second column and so on. That is, the fraction under question is the periodical fraction 0.(1) = 0.(11) = …, which is positioned in the universal table indefinitely far from the beginning and ***which is identified according to the rule* (3) *with the natural number* 1 *(one)***. So, under the used manner of numbering of dyadic fractions ***the diagonal method of Cantor runs idle***: instead of the desired non-numbered fraction it results in the integer number 1. The ‘number’ of the corresponding column is ***virtually*** the ‘greatest’ natural number 111…1…, which may be symbolized as (1) = (11) = … [999…9… = (9) in decimal calculus]. This virtual number could be identified in the present (Cantor’s) hierarchy of the transfinite numbers as the smallest ordinal $\omega$.

In the binary number system non-reducible fractions *p/q*, in which denominators *q* contain odd prime multiples, are also located infinitely far in the universal table. They correspond to periodic or combined periodic fractions of the type 0.*a*(*b*), where *a*, *b* are finite natural numbers. The numbers *a*(*b*) of these fractions are already infinite. So, infinity in this situation appears as the result of conventional use of the positional systems of calculation in arithmetic.

We note further that each column of the table 1 may be put into one-to one correspondence not only with its number $a \equiv a_0a_1a_2\ldots a_i\ldots$ or the conforming dyadic fraction 0.*a* (see eqs. (1), (2)), but also with some subset of the natural series $\{i_1(a), i_2(a), \ldots\}$, in which the index $i_k$ runs over the indices of equal to 1 constituents $a_i$ of the sequence $a_0a_1a_2\ldots a_i\ldots$. For instance, the 11-th column of the table corresponds to its dyadic number 1101, the fraction 0.1101 and a subset {0, 1, 3} of the natural series.

So, by means of the direct construction the one-to-one correspondence is established between three types of sets: infinite series (natural number sequence), the set of all dyadic (and obviously all *r*-adic) fractions and the set of all subsets of infinite series. We note at once that such correspondence between natural series and the set of all its subsets contradicts to the famous ***Cantor’s theorem***, which in essence reminds his ***diagonal method*** and also may be called in question.

The number of elements (power, cardinality) of the set of all subsets of the finite set of the order *n* is known to be equal to $2^n$. It is natural to designate the cardinality of the set of all subsets of the natural number series as $2^{\aleph_0}$. According to Cantor, $2^{\aleph_0} = c$. Along with the inequality $c > \aleph_0$ this leads to the chain of cardinal numbers

$$\aleph_0 < c < 2^c < \ldots , \qquad (6)$$

which is somewhat more meaningful than the “misty” chain of alephs (3) containing the chain (6). The proposal that two chains (3) and (6) coincide is called the ***generalized continuum hypothesis***. The statement $c = \aleph_0$ converges ***all*** infinite cardinal numbers to the single one – some conditional designation of the Aristotelian ***potential infinity*** ($\infty$ in usual practice). However, this statement can be confidently applied only to the Cantor’s sets with distinguishable elements. Evidently, the ‘whole’ continuum is a non-Cantorian set, while everywhere dense set of rational fractions and a set of algebraic numbers are Cantorian. The pointed sets of numbers differ not by their cardinality, but by their continuity.

Different monotonically growing functions of the type $n^2$, $2^n$, $n!$, lg $n$ are distinguished by the velocity of their "perpetual motion" to the infinity. This allows to speak about infinitely greats (may be, it is better to say *arbitrarily (indefinitely?) greats*, AG) of the type $O(n^2)$, $O(2^n)$, $O(n!)$, $O(\lg n)$. The comparison of sequences $(n)$, $(n^2)$ and $(2^n)$ testify to the equality of their cardinalities, though in Cantorian conception the exponential function definitely stands apart.

Similarly, the infinitesimals (*arbitrary smalls*) of the type $o(1/n)$, $o(1/2^n)$, $o(1/n!)$ and so on are related to the infinite fragmentation of the unit. So, some symmetry can be established between zero and infinity: infinity is somewhat inaccessible, zero is something which is not existent, naught. ***Zero, as a result of infinite fragmentation of unit, is inaccessible as much as infinity***. So, it exists only virtually.

The notion 'unit' in the science is used, first of all, as a unit of measurement. The units of measurement in standard system are close to the human being parameters – kilogram, meter, second. In other systems units may differ from standard ones by many millions times. As units for measuring of angles the degrees and radians are used, 180 degrees being equal to $\pi$ radians. Therefore. the unit in principle may be considered as a ***representative*** of an arbitrary finite real number.

Zero, unit and appearing in the process of infinite succession integer numbers constitute the natural number series, "given by God" as said by L. Kroneker.

## Conclusion.

As it was mentioned above the introduction of the set theory into mathematics as its foundation was accompanied by lively discussions. The enthusiastic comments of D. Hilbert and B. Russell alternated with critical statements by A. Poincare an G. Weyl. Poincare wrote in 1908: "Later generations will regard set theory as a disease from which one has recovered". The fact, that the set theory along with the Cantor's concept of infinity took the firm position in mathematics, may be accompanied by the remark of J. von Neumann to F. Smith: "Young man, in mathematics you don't understand things. You just get used to them". Or, as M. Plank once said referring to the quantum mechanics: "A scientific truth does not triumph by convincing its opponents and making them see the light, but rather because its opponents eventually die, and a new generation grows up that is familiar with it." However, in the process of development of science it is natural to return from time to time to discussion moments of its history.

The mathematics, as A. Einstein wrote in 1921, "owes its existence to the need … of learning something about the behavior of real objects". "As far as the propositions of mathematics refer to reality, they are not certain; and as far as they are certain, they do not refer to reality". Indeed, the very processes of picking out of objects under study from surroundings and measurement influence somehow the results of investigation. The precision of measurements is always finite. The growth of precision is 'supplemented' by the growth of complexity of apparatus. So, in view of possible applications in natural sciences, infinity and infinite precision are accessible only virtually, in mind. An existence of different stages of infinity, in fact postulated by Cantor, seems to be not sufficiently grounded. Besides, some questions connected with the choice of systems of axioms arise.

In applied mathematics there is no necessity to structure infinity by cardinalities. The natural series is great enough, and any finite number series as great as desired may be considered as the very beginning of it. It is great enough as to digitize all sciences of the present and future.

As the practical problems are solved, what may be considered as the steps of the building of science, the necessity to take care of foundation – logic, set theory – arises. This produces extra emotional background for scientists which is well described by the following citation from the essay “Mathematics and logic” by G. Weyl (1946):

“From this historical essay the following is quite clear: we all less and less believe to the presence of sufficient grounds of logic and mathematics. As all in the modern world we have our own ‘crisis’. And it lasts already about half century. ***From outside it is hardly noticeable that this crisis interferes our everyday work***; however I myself, for instance, must confess that it leaves the deep mark on all my mathematical work … It constantly damps enthusiasm and resolution with which I set to my scientific researches.”

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